

REPORT DOCUMENTATION PAGE

AFRL-SR-BL-TR-00-

Public reporting burden for this collection of information is estimated to average 1 hour per response, including the time for reviewing instructions, searching existing data sources, gathering and maintaining the data needed, and completing and reviewing this collection of information. Send comments regarding this burden estimate or any other aspect of this collection of information, including suggestions for reducing the burden, to Washington Headquarters Services, Directorate for Information Operations and Reports, 1215 Jefferson Davis Highway, Suite 1204, Arlington, VA 22202-4302, and to the Office of Management and Budget, Paperwork Project, Washington, DC 20503.

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e for Information

1. AGENCY USE ONLY (Leave blank)		2. REPORT DATE	3. 01 Jul 96 to 30 Jun 99Final
4. TITLE AND SUBTITLE (AASERT95) Transport Processes in Plasmas			5. FUNDING NUMBERS 61103D 3484/TS
6. AUTHOR(S) Professor Lebowitz			
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES) Rutgers, The State University of New Jersey ASB Annex II PO Box 1179-Busch Piscataway NJ 08855-1179			8. PERFORMING ORGANIZATION REPORT NUMBER
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) AFOSR/NE 801 North Randolph Street Rm 732 Arlington, VA 22203-1977			10. SPONSORING/MONITORING AGENCY REPORT NUMBER F49620-96-1-0281
11. SUPPLEMENTARY NOTES			
12a. DISTRIBUTION AVAILABILITY STATEMENT APPROVAL FOR PUBLIC RELEASED; DISTRIBUTION UNLIMITED			12b. DISTRIBUTION CODE
13. ABSTRACT (Maximum 200 words) The AASERT Grant supported two students at Rutgers University, Mr. Dov Chelst and Mr. Kevin Rosema. Dov Chelst received his Ph.D. in May 1999. The title of his thesis was "Modified Two Component Plasmas and Generalizations of Schwarz's Lemma". Dov's thesis dealt with statistical mechanics of one dimensional systems with Coulomb interactions. Mr. Kevin Rosema was supported by this grant for the period of one year. He is currently continuing his graduate studies and expect to finish at the end of this year. His work on the grant involved fluctuations in Coulombic type systems.			
14. SUBJECT TERMS			15. NUMBER OF PAGES
20000208 031			16. PRICE CODE
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL

Final Technical Report

AASERT Grant F49620-96-1-0281

July 1, 1996 – June 30, 1999

The AASERT Grant supported two students at Rutgers University, Mr. Dov Chelst and Mr. Kevin Rosema.

Dov Chelst received his Ph.D. in May 1999. The title of his thesis was "Modified Two Component Plasmas and Generalizations of Schwarz's Lemma". Dov's thesis dealt with statistical mechanics of one dimensional systems with Coulomb interactions. This is a problem on which some beautiful analytic work was done a long time ago by Baxter, Edwards, Lenard, and others. Their solutions employed an implicit equation involving a continued fraction. In addition, Lenard used the analytic theory of continued fractions to prove the existence of an implicit solution. However, he did not realize that this theory could also be used to prove the *analyticity* of this solution with the aid of an implicit function theorem.

Chelst thesis extends Lenard's analysis to include a large class of systems so called modified two component plasmas. This includes both Coulombic and non-Coulombic types of interaction.

The motivation for this arose from some experimental, numerical and approximate analytical calculations in higher dimension. These indicate that Coulomb systems with hard cores undergo a rather peculiar kind of liquid-vapor transition. It seemed that it would be useful if one could investigate the simpler one dimensional system exactly. This Dov has done exceptionally well, adding both short range interactions as well as a long

range Kac potential, which induces a liquid-vapor type phase transition in such systems. He was then able to show that this transition did not show peculiar behavior in one dimension.

Mr. Chelst has already published one paper on his results (copy attached). Other papers resulting from the thesis are now in preparation. A copy of Mr. Chelst's thesis and publication will be sent separately. At the present time Dr. Chelst has a teaching position at Rutgers University.

Mr. Kevin Rosema was supported by this grant for the period of one year. He is currently continuing his graduate studies and expect to finish at the end of this year. His work on the grant involved fluctuations in Coulombic type systems. In particular he developed several computer program to obtain systems whose fluctuations are subnormal, i.e. grow slower than the volume. This is a characteristic of Coulomb systems and possibly other related systems. This work has not been published.

Appendix A: Three copies of the manuscript: Absence of Phase Transitions in Modified Two-Component Plasmas: The Analytic Theory of Continued Fractions in Statistical Mechanics, by Dov Chelst.

Appendix B: Modified Two Component Plasmas and Generalizations of Schwarz's Lemma, by Dov Chelst, Ph.D. Thesis

Absence of Phase Transitions in Modified Two-Component Plasmas: The Analytic Theory of Continued Fractions in Statistical Mechanics

Dov Chelst

ABSTRACT. In 1961, A. Lenard[9] and S. Prager[11] independently solved the one-dimensional two-component plasma. Their solutions employed an implicit equation involving a continued fraction. In addition, Lenard used the analytic theory of continued fractions to prove the existence of an implicit solution. However, he did not realize that this theory could also be used to prove the *analyticity* of this solution with the aid of an implicit function theorem.

We have extended Lenard's analysis in [3] to include a class of systems which we call **modified two-component plasmas**. Any such system can still be described in terms of an implicit continued fraction equation. In this paper, we intend to show that such a solution is analytic. Thermodynamically, this implies the non-existence of pressure-dependent phase transitions, i.e. transitions between two phases that occur as pressure P and density ρ vary while the inverse temperature β remains fixed.

1. Introduction: Statistical Mechanics, Phase Transitions and One-Dimensional Systems

It is not our intent to develop new theorems regarding the convergence or analyticity of continued fractions. Rather, we intend to apply well-known theorems (Theorems 6 and 7) to tackle a problem in statistical mechanics. Thus, this paper will contain no new results for continued fractions, just a new application. However, we believe this application to be quite significant.

What is statistical mechanics? Statistical mechanics studies systems containing a large number of particles. These particles obey a set of microscopic dynamical laws. Starting from *microscopic* particle interactions, such as pair potentials, one seeks to derive expressions for thermodynamic *macroscopic* quantities such as the pressure P , as functions of the inverse temperature β and particle density ρ . Within

1991 *Mathematics Subject Classification*. Primary 82B23, 82D10; Secondary 82B05, 82B26, 30B70.

This work was partially supported by AASERT Grant No. AS0281.

I would like to thank the organizers of this conference for allowing me to communicate my ideas to members of the continued fractions community.

I would also like to thank Prof. Eugene Speer for discussing the details of this paper's main argument with me and for scrutinizing my earlier drafts.

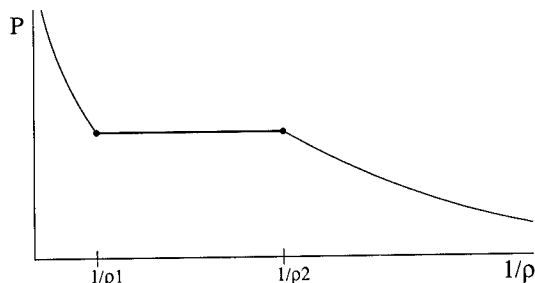
this theory, these thermodynamic quantities correspond to either averages or sums over a tremendous number (e.g. 10^{20}) of microscopic variables. The large number of particles is responsible for the statistical nature of the theory.

While developing a thermodynamic portrait of a given system, one naturally asks if it exhibits a phase transition. A collection of particles can behave in different ways: like individual molecules (a gaseous state), like a loosely bound conglomerate (a liquid state), or like a more rigidly bound molecular lattice (a solid state). The question yet remains: for a given system, do clear demarcations exist between these different states? Do they represent distinct phases of matter?

A phase transition is characterized by an abrupt change. For example, when we increase the pressure on a gas, while maintaining its temperature, its molecules may condense to form a liquid. Thus, at a specific pressure, a sudden change in density would occur. Up to this point, the pressure and density would continuously increase upon the gas.

To describe this behavior mathematically, we have a fundamental thermodynamic relation which describes the system as a function of a "complete set" of thermodynamic variables. One example of such a relation is the average entropy per particle $s(e, \rho)$ which depends upon the mean energy per particle e and the particle density ρ of a system. Another formulation, the chemical potential (or Gibbs potential) $\mu(P, \beta)$, which gives rise to this article's continued fraction, depends explicitly upon a system's pressure P and inverse temperature β . All other thermodynamic quantities are calculated as partial derivatives of this fundamental relation. For example, the density ρ of a system for some P and β is related to μ by

$$(1) \quad \frac{1}{\rho} = \frac{\partial \mu(P, \beta)}{\partial P}.$$



A phase transition can be defined as a discontinuity in one of μ 's derivatives. For example, in the above diagram, a liquid-vapor phase transition of van der Waals type can be described by the non-differentiability of a pressure-density curve (isotherm). Below a critical temperature, there is a certain pressure which corresponds to two distinct densities ρ_1 and ρ_2 , representing matter in two different phases. For any intermediate value of the density, each phase occupies a proportional fraction of the system's total volume. At either end of this interval along the isotherm, the curve has a discontinuity in its first derivative, i.e. a first-order phase transition.

While one can often prove that a physical system exhibits thermodynamic behavior, one can rarely calculate its fundamental relation exactly based solely upon the knowledge of its microscopic interactions. Thus, to obtain exact results, one needs to examine idealized models. One such idealization involves restricting

the analysis to one rather than three dimensions. In one dimension, a number of models have exact solutions.

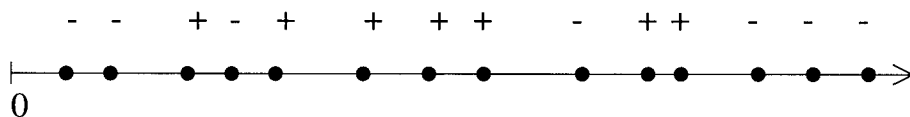
Rigorous proofs for the non-existence of phase transitions are another rare commodity in statistical mechanics. These theorems, when they exist, focus on showing that some fundamental relation describing a given system is *analytic*. In one-dimension, the most famous non-existence proof certainly follows this line of reasoning. In 1950, Van Hove[5] showed that one-dimensional systems with only finite-neighbor interactions exhibit no phase transitions.

Plasmas, which consist of charged particles, interact via a *long-range* Coulomb interaction and are not covered by Van Hove's theorem. Nevertheless, in 1962, Lenard and S.F. Edwards[4] showed, by analyzing the eigenvalues of a differential equation of Mathieu type, that the one-dimensional two-component plasma can exhibit no phase transitions.¹ This was proven in a way that avoided the continued fraction which featured so prominently in Lenard's original thermodynamic solution of the system.

We can readily generalize the original thermodynamic argument of [9] to include modified two-component plasmas.² One simply notes that Lenard and Prager's original argument uses a Laplace transform technique reminiscent of the solutions of Tonks and Takahashi for systems with only hard-cores or with only nearest-neighbor potentials respectively.³ Then, one replaces Lenard's Laplace transform of the identity with a more complicated Laplace transform. However, it is not clear how to generalize Lenard's second argument, that proved the non-existence of phase transitions, in so broad a fashion. Luckily, this is not necessary.

2. Modified Two-Component Plasmas

When expressing the thermodynamics of this model, we fix β and the unit charge σ . Thus, in order to focus upon the crux of the argument and eliminate distracting notation, we set $\sigma = 1$ and $\beta = 1$. We therefore intend to suppress all dependence upon these variables, totally ignoring them in all but two places: at the ends of Sections 4.3 and 4.4.



Modified two-component plasmas contain an equal number of positively and negatively charged particles with unit charges $+1$ and -1 respectively. Each particle resides on the positive half line, and is specified by a pair of coordinates (σ_i, x_i) describing its charge and position. Pairs of particles interact via a one-dimensional Coulomb potential

$$u_{i,j} = -\sigma_i \sigma_j |x_j - x_i|.$$

When considered in three-dimensional space, we view these particles as parallel "charged sheets" that lie perpendicularly to the x-axis and each of which has a "charge density" σ_i .

¹In fact, he proved it for more general multi-component charged systems satisfying certain conditions.

²The entire argument can be found in [3].

³See [13, pp.48-53]. A translation of Takahashi's original article can be found in [10, p.25].

In addition, nearest-neighbor particles, separated by a distance x , interact via a potential $\psi(x)$. This potential may depend upon the relative charges of the two particles. In this case, we refer to two potentials, ψ_{same} and ψ_{opp} , as the interactions between neighboring pairs of ions with the same charges and with opposing charges respectively.

The total potential energy H of this system in a given configuration is just the sum of all the pair potentials. Using H , for a fixed pressure P , we employ a statistical ensemble especially suited to one-dimensional calculations, the isobaric-isothermal ensemble⁴, to determine the fundamental relation $\mu(P)$ exactly. That is all we will say about the basic description of modified two-component plasmas and their statistical mechanical calculations. A full derivation of $\mu(P)$ can be found in [3]. We will now proceed to the fruits of this analysis.

3. Thermodynamic Results: The Real and the Complex

The chemical potential μ of a modified one-dimensional two-component plasma is related to the implicit solution of the equation

$$(2) \quad \hat{Q}(P, z) = 1,$$

for positive P . Specifically, the implicit function $z^*(P)$ defined to be the **solution of smallest modulus** of (2) for fixed $P > 0$, is related to μ by

$$(3) \quad z^*(P) = e^{2\mu(P)}.$$

Through this relation, we will show that $\mu(P)$ is an analytic function of P .

Before we can examine z^* we need more information about \hat{Q} . \hat{Q} can be described in two ways:

1. **Power Series:** Initially, for fixed $P > 0$, we define \hat{Q} as a power series in z about 0,

$$(4) \quad \hat{Q}(P, z) = \sum_{n=1}^{\infty} \hat{Q}_n(P) z^n.$$

Each coefficient $\hat{Q}_N(P)$ is the result of a statistical calculation involving a modified two-component plasma with $2N$ particles using the isobaric-isothermal ensemble. As a consequence, each coefficient is a Laplace transform of a nonnegative function⁵, whose integral converges for all positive pressures P and hence for all complex P with positive real part. Our main concern is that it is positive for all positive P .

2. **Continued Fraction:** Later in [3], we show that after imposing a few conditions and employing a recursive analysis, a modified two-component plasma's \hat{Q} can be described by a continued fraction whose partial numerators and denominators are functions of P and z .

The first description shows us that when $P > 0$, and $z > 0$ is within the radius of convergence about 0, $\hat{Q}(P, z)$ and $\frac{\partial \hat{Q}}{\partial z}$ are positive. The second description allows us to fully utilize complex analytic machinery. With it, we can show that \hat{Q} is a meromorphic function of both P and z . We will see that \hat{Q} 's continued fraction is

⁴See Percus[8] for a discussion of this ensemble.

⁵ $\hat{Q}_N(P)$ corresponds to a portion of the isobaric-isothermal partition function and is thus a Laplace transform of some portion $\hat{Z}_N(L) > 0$ of the canonical partition function.

well-defined when P is restricted to some appropriate open set \mathcal{O}_ϵ containing the positive reals \mathbb{R}_+ , while z is allowed to range over all \mathbb{C} .

The continued fraction describing \hat{Q} can be written in the standard form

$$(5) \quad \hat{Q}(P, z) = \mathbf{K} \left(\frac{a_n}{b_n} \right).$$

When ψ is charge-dependent, the partial numerators and denominators are quite complicated:

$$(6) \quad \begin{cases} a_1 = \frac{2}{\eta_{avg}(P)} z \Theta(P+1), \\ a_n = -z \Theta(P+n^2), \\ b_0 = \frac{2}{\eta_{avg}(P)} z \Xi(P+1), \text{ and} \\ b_n = \eta_{opp}(P+n^2) - z \Xi(P+(n+1)^2). \end{cases}$$

Each η is the reciprocal of a Laplace transform:

$$\frac{1}{\eta_{same}(s)} = \mathcal{L}(e^{-\psi_{same}})(s), \quad \frac{1}{\eta_{opp}(s)} = \mathcal{L}(e^{-\psi_{opp}})(s), \text{ and } \frac{2}{\eta_{avg}(s)} = \frac{1}{\eta_{same}(s)} + \frac{1}{\eta_{opp}(s)}.$$

Ξ and Θ depend upon η_{opp} and η_{same} through the relations

$$(7) \quad \Theta(s) = \left(\frac{\eta_{opp}(s)}{\eta_{same}(s)} \right)^2, \text{ and } \Xi = \frac{1}{\eta_{opp}(s)} (1 - \Theta(s)).$$

When ψ does not depend upon the neighboring particles' charges, this description simplifies drastically. ψ_{opp} , ψ_{same} and ψ_{avg} all become equal, while $\Theta \equiv 1$ and $\Xi \equiv 0$. Hence,

$$(8) \quad \begin{cases} a_1 = \frac{2}{\eta(P)} z, \\ a_n = -z, \\ b_0 = 0, \text{ and} \\ b_n = \eta(P+n^2). \end{cases}$$

The fact that the description of these continued fractions hinges upon the reciprocal of a Laplace transform, coupled with the desire to have the continued fraction description remain valid for all positive P , motivates us to stipulate the following condition for ψ .

CONDITION 1. The Laplace transform $\mathcal{L}(e^{-\psi})(P)$ determined by any nearest neighbor potential ψ describing a modified two-component plasma will converge for all positive P and hence in the complex right half plane $\mathbb{H}_0 = \{z : \Re(z) > 0\}$.

This condition is not specific to modified plasmas; it is implicit in the standard thermodynamic description of pure nearest neighbor systems solved by Takahashi[10, p.25]. Thus, Condition 1 is imposed, implicitly or explicitly, on many systems containing nearest-neighbor potentials.

Of course, we must now convince a reader that the reciprocals of Laplace transforms, the η 's, are well-defined for appropriate values of P . In other words, we need to show that the Laplace transforms are non-zero. Only then can we show that the continued fractions given by (6) and (8) converge. We will explore the properties of Laplace transforms which make each η well-defined in Section 4.3.2.

4. The Analyticity of $\mu(P)$

4.1. The Main Result. Now, we will state our main theorem and prove it assuming Lemma 3 that we will prove in Section 4.4.

THEOREM 2. *Let a nearest-neighbor potential ψ satisfy Condition 1. When ψ is charge-dependent, let it satisfy Condition 8. Let $\hat{Q}(P, z)$ be a function that can be described both as a continued fraction (6) and as a power series (4) for fixed $P > 0$ in z about 0 with positive coefficients. Then the implicit function $\mu(P)$ defined via (2) and (3) is analytic for $P > 0$.*

4.2. Proof of Theorem 2. To prove Theorem 2, we must traverse a number of steps.

PROOF. 1. First, we state the lemma:

LEMMA 3. *If $\hat{Q}(P, z)$ satisfies the hypotheses of Theorem 2, then \hat{Q} is a **separately meromorphic function** of P and z for all $(P, z) \in \mathcal{O}_\epsilon \times \mathbb{C}$ for some appropriate open subset \mathcal{O}_ϵ of \mathbb{C} containing \mathbb{R}_+ .*

While we leave the proof of Lemma 3 to Section 4.4, we would like to describe its conclusion and give an idea of its proof here. Specifically, we show that if we fix one variable, and examine a neighborhood about any value of the other variable, there is an appropriate k th remainder of the continued fraction, $\mathbf{K}(\mathbf{a}_{n+k}/\mathbf{b}_{n+k})$, which is analytic in that neighborhood. Thus, the full continued fraction is a rational function of the first k partial numerators and denominators and the remainder term. Since these are all analytic functions of the free variable, \hat{Q} must be a meromorphic function of the free variable.

This argument utilizes two theorems from the analytic theory of continued fractions quoted in Section 4.4 as Theorems 6 and 7. At the same time, it relies heavily upon the **asymptotic behavior** of Laplace transforms described in Section 4.3. It is not surprising that for a charge-dependent system, we need an extra condition, given in Section 4.4 as Condition 8, to apply these continued fraction theorems. A charge-independent system does not require this additional condition; yet, when ψ assumes the proper form, it automatically satisfies Condition 8.

2. An extension of Hartogs' Theorem to meromorphic functions due to W. Rothstein[12] states that a separately meromorphic function, e.g. \hat{Q} , in an open complex domain, e.g. $\mathcal{O}_\epsilon \times \mathbb{C}$, is actually jointly meromorphic. Thus, \hat{Q} is **jointly analytic** in a neighborhood of any solution to (2).
3. Moreover, we note that the power series description (4) of \hat{Q} implies that $\frac{\partial \hat{Q}}{\partial z}$ is non-zero about any solution to (2) provided that both coordinates are positive and z lies in the radius of convergence of \hat{Q} . Together with the meromorphicity of \hat{Q} , this assures us that for $P > 0$, \hat{Q} 's first pole away from zero occurs on the positive real axis. It also guarantees that, for positive P , a **unique positive solution** $z(P)$ exists within this radius of convergence and that this solution has the smallest modulus of all solutions to (2). Thus, this solution is actually $z^*(P)$.
4. We may now employ the **complex analytic implicit function theorem**[6, 1.B.6] to prove that $z^*(P)$ is an analytic function of P . Let us briefly review the implicit function theorem.

THEOREM 4 (Implicit Function Theorem). *Let $f : U \rightarrow \mathbb{C}$ be a jointly analytic function in some open domain $U \subset \mathbb{C}^n$. In addition, let $f(\vec{x}, z) = 0$ for some $(\vec{x}, z) \in U$, and $\frac{\partial f}{\partial z}(\vec{x}, z) \neq 0$. Then there exists a neighborhood U' about \vec{x} and an analytic function $g : \mathbb{C}^{n-1} \rightarrow \mathbb{C}$ such that $f(\vec{x}, g(\vec{x})) = 0$ and $(\vec{x}, g(\vec{x})) \in U$ for all $\vec{x} \in U'$.*

This theorem applies to our situation when we replace f by $\hat{Q} - 1$. Since f is meromorphic, it is automatically analytic in a neighborhood of any point that is not a pole of f which includes any point where $f = 0$. Thus, by the implicit function theorem, $z^*(P)$ is analytic in an appropriate neighborhood of P .

5. Since $P > 0$ was chosen arbitrarily, $z^*(P)$ must be analytic for all positive P . The fact that the solution of smallest modulus $z^*(P)$ is always real and positive makes this global result possible. Normally, given two analytic implicit solutions $z_1(P)$ and $z_2(P)$ to a single equation, a situation might arise in which z_1 would have the smaller modulus for some values of P while z_2 would have the smaller modulus for other values of P . The fact that the solution $z^*(P)$ of smallest modulus is always positive implies that for such a switch to take place $z_1(P)$ and $z_2(P)$ must coincide for some value of P . By the assumptions of the implicit function theorem, namely that $\frac{\partial \hat{Q}}{\partial z} \neq 0$, this cannot occur.
6. The fact that $z^*(P)$ is analytic and positive allows us to conclude that $\mu(P) = \frac{1}{2} \ln(z^*(P))$ is also analytic in P .

□

4.3. Three Features of Laplace Transforms. As \hat{Q} is defined in terms of Laplace transforms, we should review their relevant features. More specifically, we list three features of Laplace transforms. The first is satisfied for any function with an absolutely convergent Laplace integral. The second is specific to transforms of nonnegative functions (that are not identically zero), which is certainly true in our case as $e^{-\psi}$ is never negative. The third holds true only for an even more specific situation; this is meant to serve as an example of a Laplace transform's leading-order asymptotics.

4.3.1. Limiting Behavior. If the Laplace transform of a function f , $\mathcal{L}(f)(s) = \int_0^\infty f(t)e^{-st}dt$ converges absolutely for some s , it is analytic and defined for all P in some open right half plane $\mathbb{H}_w = \{s : \Re(s) > w\}$. In addition, $\lim_{\Re(s) \rightarrow \infty} \mathcal{L}(f)(s) = 0$. In our situation, by the assumptions of Condition 1, $\mathcal{L}(e^{-\psi})(s)$ is defined in \mathbb{H}_0 and for $s \in \mathcal{O}_\epsilon \subset \mathbb{H}_0$, $\lim_{n \rightarrow \infty} |\eta(s + n^2)| = \infty$.

4.3.2. Transforms of Nonnegative Functions and Their Zeros. For a **nonnegative** function f that converges in \mathbb{H}_0 , this limiting behavior is monotonic for real P . In addition, for complex s , $|\eta(s)| \geq \eta(\Re(s))$. Moreover, if f is nonnegative and not identically 0, we can be sure that η is well-defined in some neighborhood \mathcal{O} containing \mathbb{R}_+ . After all, $\mathcal{L}(f)(s)$ is analytic in \mathbb{H}_0 and strictly positive on \mathbb{R}_+ .

Unfortunately, this does not suffice. It is necessary to find a complex neighborhood \mathcal{O}_ϵ , containing \mathbb{R}_+ , in which $\mathcal{L}(f)(s)$ and $\mathcal{L}(f)(s + n^2)$ are nonzero for all n . We will prove that there exists a neighborhood \mathcal{O}_ϵ containing \mathbb{R}_+ , on which $\mathcal{L}(f)$ is nonzero, and which has the property that $s \in \mathcal{O}_\epsilon$ implies $s + \delta \in \mathcal{O}_\epsilon$ for all $\delta > 0$. In other words:

LEMMA 5. Let $\mathcal{L}(f)$ be a Laplace transform of a nonnegative function f that converges for all $P \in \mathbb{H}_0$. If f is not identically zero, then the zeros of $\mathcal{L}(f)$ are bounded away from \mathbb{R}_+ in any half-plane \mathbb{H}_x for $x > 0$.

PROOF. We will show more than the lemma requires. We will show that the zeros of the real part of $\mathcal{L}(f)$ are also uniformly bounded away from the positive real axis in \mathbb{H}_x for any $x > 0$. Given the original function f and a positive real value x , pick some $\epsilon < \frac{1}{2}$. Now, by simple convergence arguments, one can choose a constant K so that $\int_0^K f(t)e^{-xt}dt > 0$ and

$$\int_K^\infty f(t)e^{-xt}dt \leq \epsilon \int_0^K f(t)e^{-xt}dt.$$

Then, for all $s_1 \geq x$,

$$\begin{aligned} \int_K^\infty f(t)e^{-s_1t}dt &\leq e^{-(s_1-x)K} \int_K^\infty f(t)e^{-xt}dt \leq \epsilon e^{-(s_1-x)K} \int_0^K f(t)e^{-xt}dt \\ &\leq \epsilon \int_0^K f(t)e^{-s_1t}dt. \end{aligned}$$

Now, simply choose $w > 0$ so that $wK \leq \frac{\pi}{3}$. For all real values $|s_2| \leq w$ and $0 \leq t \leq K$, $\cos s_2t \geq \frac{1}{2}$ and for complex $s = s_1 + is_2$ in the complex strip $\mathcal{S}_x = \{s : \Re(s) > x, |\Im(s)| < w\}$,

$$\begin{aligned} \Re\{\mathcal{L}(f)(s)\} &= \int_0^\infty f(t) \cos s_2t e^{-s_1t}dt \geq \frac{1}{2} \int_0^K f(t)e^{-s_1t}dt - \int_K^\infty f(t)e^{-s_1t}dt \\ &\geq \left(\frac{1}{2} - \epsilon\right) \int_0^K f(t)e^{-s_1t}dt > 0. \end{aligned}$$

The proof is complete when we note that since \mathcal{S}_x contains no zeros of $\mathcal{L}(f)$, the Laplace transform's zeros must all remain at least a distance $w > 0$ away from \mathbb{R}_+ in \mathbb{H}_x . \square

Before we continue, we simply construct an open set using the conclusion of the previous lemma. Given a suitable nonnegative function f and a positive $\epsilon < \frac{1}{2}$, define the "zero-less" open set $\mathcal{O}_\epsilon = \bigcup_{x>0} \mathcal{S}_x$. Thus, because $\mathcal{L}(f)$ has no zeros and is analytic in \mathcal{O}_ϵ , we are assured that η is well-defined and *analytic* there as well.

Finally, we note that this lemma completes a pair of inequalities. For any $s \in \mathcal{O}_\epsilon$,

$$(9) \quad |\mathcal{L}(f)(s)| \leq \mathcal{L}(f)(\Re(s)) \leq \frac{1+\epsilon}{\frac{1}{2}-\epsilon} |\mathcal{L}(f)(s)|.$$

4.3.3. *Leading Order Asymptotics of $\mathcal{L}(f)$ - Watson's Lemma.* While it is not completely necessary, the inequality (9) allows us to consider the *complex* asymptotic behavior of $\mathcal{L}(f)(s)$ as $\Re(s) \rightarrow \infty$. While the limit of $\mathcal{L}(f)$ is certainly an asymptotic feature, in a charge-dependent modified plasma system, where two Laplace transforms play a role, we need a clearer understanding of $\mathcal{L}(f)$'s asymptotic behavior.

Toward this end, we make a large assumption about f . Every nonnegative function has a set $f^{-1}((0, \infty)) \in \mathbb{R}_+$ of points where f is positive. For a function f that has a right-handed limit near the smallest point p in this set's closure, $\mathcal{L}(f)(s)$'s asymptotic behavior for large $\Re(s)$ depends very strongly upon this limit. Watson's

Lemma[2, 6.4] covers a number of possible limiting behaviors. While this does not exhaust all possible functions, nor does it include all physically meaningful functions⁶, we will restrict ourselves to those potentials covered by Watson's Lemma. An industrious reader will see that those generalizations of Watson's Lemma that are also found in [2] can be treated similarly.

Watson's Lemma states that if a function f behaves as At^α to leading order as $t \rightarrow 0^+$, for some constants A and $\alpha \geq 0$ and if $\mathcal{L}(f)$ converges in some \mathbb{H}_x , then the Laplace transform $\mathcal{L}(f)(s)$ decays as $\Re(s) \rightarrow \infty$, to leading order, like $\frac{A\Gamma(\alpha+1)}{\Re(s)^{\alpha+1}}$. (Γ is the Gamma function which for positive integers n satisfies $\Gamma(n+1) = n!$ and using Euler's integral can be written as $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$.) This is certainly true for real s and the statement for complex s follows from (9).

In statistical mechanics, we frequently encounter potentials ψ that contain hard core exclusions. In terms of $f = e^{-\beta\psi}$, a hard core specifies that $f(t) = 0$ for all $t < b$ for some hard core diameter b . If, in addition, as $t \rightarrow b^+$, f behaves as $A(t-b)^\alpha$ for some constants A and $\alpha \geq 0$, its Laplace transform $\mathcal{L}(f)(s)$ decays, to leading order, as $\frac{A\Gamma(\alpha+1)e^{-b\Re(s)}}{\Re(s)^{\alpha+1}}$. This follows from Watson's Lemma and an elementary feature of Laplace transforms that can be found in any undergraduate text on the subject. The reciprocal of $\mathcal{L}(f)$ in either of these two situations, $b = 0$ or $b \neq 0$, diverges to infinity in an appropriate manner.

We note that while Watson's Lemma also allows for $-1 < \alpha < 0$, a physical argument disallows these exponential values. After all, since $f = e^{-\beta\psi}$, our exponent α depends *linearly* upon the inverse temperature β . Thus, while for some values of β , $0 > \alpha = \alpha'\beta$ would be greater than -1 , there would certainly be values of β for which this would be false and thus the Laplace transform would not even exist. This contradicts Condition 1 that we imposed upon our nearest-neighbor potential ψ in Section 3.

This section on Laplace transforms provides all that we need in this paper. It furnishes analyticity criteria. It gives a complex open set in which all η 's are well-defined, i.e. just the finite intersection of open sets for each separate η . It also provides the asymptotic information regarding our η 's that allows us to apply our continued fractions theorems.

4.4. The Meromorphicity of \hat{Q} .

PROOF OF LEMMA 3. As stated in the proof of Theorem 2, our aim is to show that some remainder of the continued fraction \hat{Q} is separately analytic in $\mathcal{O}_\epsilon \times \mathbb{C}$; i.e. analytic in a single variable while the other remains fixed. For this purpose, we employ two theorems found in chapter 4 of the book by Thron and Jones[7, Thms. 4.35 and 4.54]. Note that f_n refers to the n th convergent of the continued fraction whose value we can obtain by substituting 0 for a_{n+1} .

THEOREM 6 (Pringsheim). *If $|b_n| \geq |a_n| + 1$ for all n , then*

- i. $\mathbf{K}\left(\frac{a_n}{b_n}\right)$ converges, and
- ii. $|f_n| \leq 1$ for all n .

⁶Try $e^{-\frac{K}{y^n}}$ from a potential $\psi(y) = \frac{K'}{y^n}$.

THEOREM 7. *If:*

- i. $\mathbf{a}_n(x)$, $\mathbf{b}_n(x)$ are analytic in some domain D , and
- ii. $\exists \xi_1, \xi_2 \in \mathbb{C}$, s.t. for all $x \in D$ and $n \in \mathbb{Z}^+$, $\mathbf{f}_n(x) \notin \{\xi_1, \xi_2, \infty\}$.

Then $\mathbf{K}\left(\frac{\mathbf{a}_n(x)}{\mathbf{b}_n(x)}\right)$ is analytic in D .

We first note that all the partial numerators and denominators described in (6) and (8) are analytic functions of P and z . In \mathcal{O}_ϵ , both η_{opp} and η_{same} are nonzero and hence their ratio is an analytic function. Thus, Θ and Ξ are both analytic functions of P . Analyticity in z is even more obvious.

Now, consider $\Theta(P)$ and $\Xi(P)$ defined in (7). By the asymptotic behavior of the two η 's, according to Watson's Lemma,

$$\begin{aligned} |\Theta(s)| &\sim C \frac{s_0^{2(\alpha_{opp}+1)} e^{2b_{opp}s_0}}{s_0^{2(\alpha_{same}+1)} e^{2b_{same}s_0}} = C s_0^{2(\alpha_{opp}-\alpha_{same})} e^{2(b_{opp}-b_{same})s_0}, \text{ and} \\ |\Xi(s)| &\sim D s_0^{\alpha_{opp}-2\alpha_{same}} e^{(b_{opp}-2b_{same})s_0}, \end{aligned}$$

for some constants C and D when s is in \mathcal{O}_ϵ and $s_0 = \Re(s)$.

Since $\Xi(s)$ can never grow asymptotically larger than $\Theta(s)$, if we wish to prove that \mathbf{b}_n is eventually bigger than \mathbf{a}_n we must focus on its other term, $\eta_{opp}(P+n^2)$ which grows like $A s_0^{\alpha_{opp}+1} e^{b_{opp}s_0}$ with $s_0 = \Re(P) + n^2$. For this to grow more quickly than its competing partial numerator, we impose the following condition.

CONDITION 8. Let ψ be a nearest-neighbor interaction within a modified two-component plasma. Moreover, let $f(t) = e^{-\psi(t)}$ have the form $f(t) = 0$ for $t < b$ and $f(t) \sim A(t-b)^\alpha$ as $t \rightarrow b^+$ for some nonnegative values b and α . Then, we require that

- i. $b_{opp} \leq 2b_{same}$, and
- ii. if $b_{opp} = 2b_{same}$, that $\alpha_{opp} < 2\alpha_{same}$.

If this condition holds, we are assured that eventually \mathbf{b}_n will grow asymptotically larger than \mathbf{a}_n and the conditions of Pringsheim's lemma (Lemma 6) will be satisfied. After all, whether we fix P or z , the real part of $P+n^2$, that replaces s from the previous discussion, goes off to infinity and our asymptotic analysis holds. The second lemma will automatically follow if we choose any two complex numbers ξ_1 and ξ_2 with modulus greater than 1. Consequently, $\hat{Q}(P, z)$ is meromorphic in $\mathcal{O}_\epsilon \times \mathbb{C}$. □

REMARK 9. We would like to point out that charge-independent systems do not require the additional condition stated above. If one examines their partial numerators in (8), one notices that they remain fixed as n increases. Conversely, the partial numerators grow asymptotically large regardless of the special form of $e^{-\psi}$. Thus, both lemmas provide their conclusions effortlessly.

REMARK 10. In addition, while we have not ruled out temperature-dependent phase transitions, it is possible that a similar analysis will preclude their existence as well. Paying attention to the inverse temperature β merely involves replacing $P+n^2$ by $\beta(P+n^2)$ in η 's arguments and switching the transformed function from $e^{-\psi}$ to $e^{-\beta\psi}$. The dependence upon β that arises from multiplying the arguments of η by β can be treated in exactly the same manner that we treated P above. Only

the β which figures in the *definition* of the transformed function itself $e^{-\beta\psi}$ could prove problematic.

In certain simple instances, e.g., when $\psi \equiv 0$ or when ψ is a pure hard core exclusion, temperature-dependent phase transitions are also absent. In those situations, $\mathcal{L}(e^{-\psi})$ depends on β only through its argument $\beta(P + n^2)$. Thus, in these instances, the line of reasoning described in this paper applies to β as well.

5. Conclusion

We have accomplished our goal of utilizing the theory of continued fraction to prove a fact regarding modified two-component plasmas. Of course, some questions remain unresolved. Did we use this theory effectively? The theorems employed provided sufficient conditions preventing the existence of phase transitions in these systems. But, how necessary is the additional condition imposed upon plasmas with charge-dependent nearest neighbor interactions. What happens if $b_{opp} > 2b_{same}$?

Continued fractions may yet hold the answer to this question. Continued fractions continue to persist in many models related to the two-component plasma. They appear explicitly in the discussion of one-dimensional ion-dipole systems in [15]. Moreover, they are implicit in every article that calls upon the reader to calculate an eigenvalue of a Mathieu-type differential (differential-delay) equation with periodic boundary values.⁷ For these reasons, we believe that continued fractions will continue to provide valuable information regarding this family of statistical models.

References

- [1] R.J. Baxter. One-dimensional gases with hard-core repulsion. *The Physics of Fluids*, 8(4):687–692, 1965.
- [2] Carl M. Bender and Steven A. Orszag. *Advanced Mathematical Methods for Scientists and Engineers*. McGraw-Hill, 1978.
- [3] Dov Chelst. *Modified One-Dimensional Two-Component Plasmas and Generalizations of Schwarz's Lemma*. PhD thesis, Rutgers University, 1999.
- [4] S.F. Edwards and A. Lenard. Exact statistical mechanics of a one-dimensional system with coulomb forces. ii. the method of functional integration. *Journal of Mathematical Physics*, 3(4):778–790, 1962.
- [5] Harry L. Frisch and Joel L. Lebowitz, editors. *The Equilibrium Theory of Classical Fluids*, volume 15 of *Frontiers in Physics*, chapter I-7. W.A. Benjamin, New York, NY, 1964.
- [6] Robert C. Gunning and Hugo Rossi. *Analytic Functions of Several Complex Variables*. Prentice-Hall, Englewood Cliffs, NJ, 3 edition, 1965.
- [7] William B. Jones and W.J. Thron. *Continued Fractions: Analytic Theory and Applications*, volume 11 of *Encyclopedia of Mathematics and its Applications*. Addison-Wesley, Reading, MA, 1980.
- [8] J.L. Lebowitz, editor. *Simple Models of Equilibrium and Nonequilibrium Phenomena*, chapter 1. Elsevier, Amsterdam, 1987.
- [9] A. Lenard. Exact statistical mechanics of a one-dimensional system with coulomb forces. *Journal of Mathematical Physics*, 2(5):682–693, 1961.
- [10] Daniel Mattis. *The Many-Body Problem: An Encyclopedia of Exactly Solved Models in One Dimension*. World Scientific, River Edge, NJ, 1993.
- [11] S. Prager. The one-dimensional plasma. *Advances in Chemical Physics*, IV:201+, 1961.
- [12] Wolfgang Rothstein. Ein neuer Beweis des Hartogsschen Hauptsatzes und seine Ausdehnung auf meromorphe Funktionen. *Mathematische Zeitschrift*, 53:84–95, 1950.
- [13] Colin J. Thompson. *Classical Equilibrium Statistical Mechanics*. Oxford University Press, London, 1988.

⁷See [14, 15] and [1] for example.

- [14] Fernando Vericat and Lesser Blum. Statistical mechanics of ion-dipole mixtures: An exactly solvable model in one dimension. *Journal of Chemical Physics*, 82(3):1492-5, 1985.
- [15] Fernando Vericat and Lesser Blum. Many-body functions of nonprimitive electrolytes in one dimension. *Journal of Statistical Physics*, 61(5/6):1161-1185, 1990.

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